SOME THEOREMS ON THE SUMMATION OF DIVERGENT SERIES

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY, IN THE FACULTY OF PURE SCIENCE, COLUMBIA UNIVERSITY.

BY

GLENN JAMES

PRINTED BY

W. D. GRAY, 227 West 17th Street

NEW YORK CITY

1917

3781
.172J
UNCLASSED
Dissertation

STANFORD UNIVERSITY LIBRARIES
WITHDRAWN

SOME THEOREMS ON THE SUMMATION OF DIVERGENT SERIES

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY, IN THE FACULTY OF PURE SCIENCE, COLUMBIA UNIVERSITY.

BY

GLENN JAMES

PRINTED BY

W. D. GRAY, 227 West 17th Street

NEW YORK CITY

1917

SOME THEOREMS ON THE SUMMATION OF DIVERGENT SERIES

by

GLENN JAMES.

---

1. **The Definition of the Sum of a Divergent Series.**
The various definitions of the sum of a divergent series are based directly or indirectly upon the use of so called convergence factors.* A series $\sum_{1}^{n} a_p$ is transformed into

$$(1) \qquad \sum_{p=1}^{n} a_p f_p(x_1, x_2, \ldots, x_k)$$

where $f_p(x_1, x_2, \ldots, x_k)$ is such that the "subsidiary sequence," (1), converges for all values of $x_1, x_2, \ldots, x_k$ on certain specified ranges. The sum of $\sum_{p=1}^{\infty} a_p$ is then defined to be †

$$(2) \quad S = \lim_{x_1 \to L_1} \lim_{x_2 \to L_2} \cdots \lim_{x_k \to L_k} \lim_{n \to \infty} \sum_{p=1}^{n} a_p f_p(x_1, x_2, \ldots, x_k)$$

For brevity we will write the above as follows:

$$(3) \quad S = \lim_{x_i \to L_i} \lim_{n \to \infty} \sum_{1}^{n}, a_p f_p(x_i), \ (i = 1, 2, 3, \ldots, k)$$

For example the series

$$1 - 1 + 1 - 1 + \ldots$$

can be summed by taking,

$$f_p(x_i) = \frac{1}{c^{px}}, \qquad x > 0, c > 1$$

Then

$$S = \lim_{x \to \infty} \lim_{n \to \infty} \sum_{1}^{n} \frac{(-1)^{p+1}}{c^{px}} = \lim_{x \to \infty} \frac{1}{c^x + 1} = 0$$

We might, however, choose

---

* C. N. Moore: Trans. Am. Math. Soc. (1901), p. 300.
S. Chapman: Quarterly Journal (1912), V. 43, p. 1.
L. L. Smail: Dissertation, "Some Generalizations on the Theory of Summable Series," Columbia University (1913), Chap. II, p. 4.

† We may have $x_k = n$ as in Cesaro's mean, § 3. In this case there would be only k limiting processes.

$$f_p(x_i) = \frac{1}{c^{p/x}}, \qquad x > 0, c > 1$$

In this case $S = \frac{1}{2}$.

Evidently the usefulness of the general definition depends upon the restrictions which are placed upon $f_p(x_i)$.

(a) Consistency in the general theory of series demands that any method of summing divergent series should be such as to give the ordinary sum when applied to convergent series. This is called the "condition of consistency." * (It is evident that the first method in the above example does not satisfy this condition, since $\lim_{x \to \infty} \frac{1}{c^{px}} = 0$. But the second method does since,

$\lim_{x \to \infty} \frac{1}{c^{p/x}} = 1$; $\lim_{p \to \infty} \frac{1}{c^{p/x}} = 0$ and $\frac{1}{c^{p/x}}$ is always positive.) †

(b) Evidently the sum of any series should be the function whose expansion gives the series, when such a function exists uniquely.‡

(c) When we are summing a series of variable terms, which converges uniformly, it is desirable that the "subsidiary sequence" should converge uniformly with respect to the variable of the original series. If such be the case, a series is said to be "uniformly § summable" by the given method.

In general it seems desirable to seek methods of summing divergent series which make the sum bear the closest possible analogy to the ordinary sum of convergent series. However, in this paper we shall be concerned only with the above restrictions, which are indeed most essential.

In the next article a mean for summing divergent series is developed which satisfies the condition of consistency and sums uniformly convergent series uniformly. This mean includes the most interesting form of Euler's Transformation for slowly convergent series ¶ as a special case, and with a slight modification becomes a generalization of the means of Frohenins and Holder.

Properly divergent series have been shown to be non-summable by the well known methods of Borel, LeRoy, Cesaro or Hölder. || The third article of this paper extends the notion of proper divergence to a class of oscillating series which are non-summable by the above methods and, in fact, by the above general definition of p. 1, when certain restrictions are placed upon it. These restrictions

---

* Bromwich: "Theory of Infinite Series," p. 269.

† See article 3, Theorem VIII.

‡ Pringsheim: Encyklopedie, Bd. I, A. 3, 39.

§ Any series is said to be "uniformly summable" when the subsidiary sequence converges uniformly; see Hardy, Transactions Cambridge Philos. Soc., Vol. 19 (1904), p. 301.

¶ Bromwich: Loc. cit., p. 55.

|| L. L. Smail: Dissertation, Columbia University (1913), Chap. III.

are shown to make the definition satisfy the condition of consistency and to sum uniformly convergent series uniformly.

2. **A Mean for Summing Series.**

Consider the series

$$a_1 + a_2 + a_3 + \ldots + a_r = s_r \tag{4}$$

where

$$\lim_{r \to \infty} s_r = s$$

The plan followed in the first part of the succeeding development is as follows: Take the means m at a time, of the terms of (4) beginning with the first, then the second, and so on; make corrections to maintain the identity; repeat the process on the new series n–1 successive times and take the limit as r increases indefinitely.

For $n \geqq 1$ and $r \geqq m$,* we have,

$$\frac{a_1+a_2+a_3+\ldots+a_m}{m} + \frac{a_2+a_3+a_4+\ldots+a_{m+1}}{m} + \frac{a_3+a_4+a_5+\ldots+a_{m+2}}{m} + \ldots + \frac{a_{r-m+1}+a_{r-m+2}+a_{r-m+3}+a_r}{m} + \frac{(m-1)a_r+(m-2)a_{r-1}+\ldots+a_{r-m+2}}{m} = s_r - \frac{(m-1)a_1+(m-2)a_2+\ldots+a_{m-1}}{m} \tag{5}$$

The sum of the first m terms of (5) is,

$$(a_1 + 2a_2 + 3a_3 + \ldots + m\,a_m + \ldots + a_{2m-1})/m$$

From the expansion, by actual multiplication, of

$$(1 + x + x^2 + \ldots + x^{m-1})^2$$

we see that the above numerator is just this expansion with x^q replaced by a_{q+1}, $q = 0, \ldots, (2m-2)$. Hence we will denote

$$(a_1 + 2a_2 + 3a_3 + \ldots + ma_m + - - - + a_{2m-1})$$

by

$$^2(a_1 + \ldots + a_m)$$

Taking the means of (5) we now obtain, with this notation:

$$\frac{^2(a_1+\ldots+a_m)}{m^2} + \frac{^2(a_2+\ldots+a_{m+2})}{m^2} + - - - + \frac{^2(a_{r-2(m-1)}+\ldots+a_{r-m+1})}{m^2} \tag{6}$$

* Although this and the following identities hold for m=1, this is a trivial case. Hence it will be understood throughout this paper that $m > 1$.

$$+\frac{(m-1)a_r+(m-2)a_{r-1}+\ldots+a_{r-m+2}}{m}+$$

$$\frac{(m-1)(a_r+\ldots+a_{r-m+1})+(m-2)(a_{r-1}+\ldots+a_{r-m})+(a_{r-m+2}+\ldots+a_{r-2m+3})}{m^2}$$

$$=s_r-\left[\frac{(m-1)a_1+(m-2)a_2+\ldots+a_{m-1}}{m}+\right.$$

$$\left.\frac{(m-1)(a_1+\ldots+a_m)+(m-2)(a_2+\ldots+a_{m+1})+\ldots+(a_{m-1}+\ldots+a_{2m-2})}{m^2}\right]$$

Evidently the sum, as before, of the first m terms of (6), r being chosen sufficiently large, will give a form in $a_1, a_2 \ldots, a_{3m-2}$ which is just $(1+x+\ldots+x^{m-1})^3$ with x^q replaced by a_{q+1}, (q=0, 1, 2 ... 3m—3). Analogous to the procedure above we denote this form by ${}^3(a_1+\ldots+a_m)$. Similarly the nth mean will give us forms ${}^n(a_1+\ldots+a_m)$, which are equivalent to $(1+x+\ldots+x^{m-1})^n$ with $x^q=a_{q+1}$, q=0, 1, ..., n(m—1) since there are n(m—1)+1 terms in the nth power expansion of the sum of m terms.

We can now write out the identity resulting from the n—1th repetition of the above process. It is as follows:

(7)
$$\frac{{}^n(a_1+\ldots+a_m)}{m^n}+\frac{{}^n(a_2+\ldots+a_{m+1})}{m^n}$$

$$+\ldots+\frac{{}^n(a_{r-n(m-1)}+\ldots+a_{r-n(m-1)+m-1})}{m^n}+R'_r$$

$$=s_r-\left[\frac{(m-1)a_1+(m-2)a_2+\ldots+a_{m-1}}{m}+\right.$$

$$\frac{(m-1)(a_1+\ldots+a_m)+(m-2)(a_2+\ldots+a_{m+1})+\ldots+(a_{m-1}+\ldots+a_{2m-2})}{m^2}$$

$$+\ldots+\frac{(m-1)^{n-1}(a_1+\ldots+a_m)+(m-2)^{n-1}(a_2+\ldots+a_{m+1})+}{m^n}$$

$$\left.\frac{\ldots+{}^{n-1}(a_{m-1}+\ldots+a_{2m-2})}{}\right]$$

where,

$$R'_r = \frac{(m-1)a_r+(m-2)a_{r-1}+\ldots+a_{r-m+2}}{m} + \frac{(m-1)(a_r+\ldots+a_{r-m+1})+(m-2)(a_{r-1}+\ldots+a_{r-m})+\ldots+(a_{r-m+2}+\ldots+a_{r-2m+2})}{m^2}$$

$$+\ldots+\frac{(m-1)^{n-1}(a_r+\ldots+a_{r-m+1})+(m-2)^{n-1}(a_{r-1}+\ldots+a_{r-m})+\ldots+^{n-1}(a_{r-m+2}+\ldots+a_{r-2m+3})}{m^n}$$

and $$r \geq n(m-1)+1.$$

We will now seek the limiting form of (7), as r increases.

Consider first the remainder R'_r.

Since the series (4) converges, for every ϵ, there exists an r' such that for $r \geq r'$

$$\epsilon \geq |a_{r-i}|, \quad [i=0, 1, 2, \ldots, (n-1)(m-1)+m-2]$$

whence for $r \geq r'$

$$R'_r \leq \epsilon \left[\frac{(m-1)+(m-2)+\ldots+1}{m} + \frac{(m-1)m+(m-2)m+\ldots+m}{m^2} + \ldots + \frac{(m-1)m^{n-1}+(m-2)m^{n-1}+\ldots+m^{n-1}}{m^n} \right] = \epsilon \frac{n(m-1)}{2}.$$

and it follows that

(8) $\lim_{r\to\infty} R'_r = 0$, for every n and m.

The terms $\frac{^n(a_1+\ldots+a_m)}{m^n}, \frac{^n(a_2+\ldots+a_{m+1})}{m^n}, \ldots \frac{^n(a_{r-n(m-1)}+\ldots+a_{r-(n-1)(m-1)})}{m^n},$

when added term by term take the form

(9) $^n[(a_1+\ldots+a_{r-n(m-1)})+(a_2+\ldots+a_{r-n(m-1)+1})+\ldots+(a_m+\ldots+a_{r-(n-1)(m-1)})]1/m^n$. This can be written,

(10) $^n[s_{r-n(m-1)}+(s_{r-n(m-1)+1}-s_1)+\ldots+(s_{r-(n-1)(m-1)}-s_{m-1})]1/m^n$

The last term in the expansion of (10) would be

$$s_{r-n(m-1),n(m-1)+1}-s_{n(m-1)}=s_{r+1}-s_{n(m-1)}$$

Now since $\sum_1^\infty a_n = s$

$$(11)\quad \lim_{r\to\infty} s_{r-n(m-1)} = \lim_{r\to\infty} s_{r-(m-1)+1} = \ldots = \lim_{r\to\infty} s_r = \lim_{r\to\infty} s_{r+1} = s,$$

m and n being fixed.

Whence

$$(12)\quad \lim_{r\to\infty} {}^n[s_{r-n(m-1)} + (s_{r-n(m-1)+1} - s_1) + \ldots + (s_{r-(n-1)(m-1)} - s_{m-1})]\,1/m^n$$
$$= {}^n[s + (s - s_1) + \ldots + (s - s_{m-1})]\,1/m^n$$
$$= {}^n(R_o + R_1 + \ldots + R_{m-1})\,1/m^n$$

where $R_n = s$ and R_t is the remainder after the ith term.

From (8), (12) and the fact that $\lim_{r\to\infty} s_r = s$, we see that the limiting form of (7) is

$$(13)\quad {}^n(R_o + \ldots + R_{m-1})\,1/m^n = s - \left[\frac{(m-1)a_1 + \ldots + a_{m-1}}{m} + \sum_{i=1}^{n-1} \frac{(m-1)\,{}^i(a_1 + \ldots + a_{m-1}) + (m-2)\,{}^i(a_2 + \ldots + a_m) + \ldots + {}^i(a_{m-1} + \ldots + a_{2m-2})}{m^{i+1}}\right]$$

or

$$(14)\quad {}^n(R_o + \ldots + R_{m-1})\,1/m^n = s - {}_mS'_n$$

where ${}_mS'_n$ denotes the second term of the right member of (13).

Theorem I. When the series $a_1 + a_2 + \ldots + a_n + \ldots$ converges to s, $\lim_{n\to\infty} {}_mS'_n = s$

To prove the theorem, it is sufficient to show that the left member of (14) converges to zero.

Expand ${}^n(R_o + \ldots + R_{m-1})\,1/m^n$, denoting the coefficients by $c_o, c_1, \ldots, c_{n(m-1)}$ and write the expansion as follows:

$$(15)\quad \frac{R_o + c_1R_1 + \ldots + c_kR_k}{m^n} + \frac{c_{k+1}R_{k+1} + \ldots + R_{n(m-1)}}{m^n},$$

noting that $c_o = c_{n(m-1)} = 1$.

Now for every ϵ there exists a (k) such that

$$(16)\quad |R_{k+1}| \leqq \epsilon/2, \qquad i = 1, 2, 3, \ldots$$

since $\sum_1^\infty a_n = s$.

Whence

$$(17)\quad |c_{k+1}R_{k+1} + \ldots + R_{n(m-1)}|\,1/m^n \leqq \epsilon/2[c_{k+1} + \ldots + c_{n(m-1)-1} +]\,1/m^n \equiv \epsilon/2,\ n(m-1) > k$$

Consider the first k+1 terms of (15).

Since $\lim_{n\to\infty} 1/m^n = \lim_{n\to\infty} c_1/m^n = \ldots = \lim_{n\to\infty} c_k/m^n = 0$

there exists an N such that

$$(18) \qquad \frac{|R_0+c_1R_1+\ldots+c_kR_k|}{m^n} \leqq \epsilon/2, \qquad n>N.$$

Combining (17) and (18) we have

(19) $|^n(R_0+\ldots+R_{m-1})1/m^n| \leqq \epsilon$, $n>N'$, N' being equal to or greater than either $\sum_{m=1}^{k}$ or N. Thus the theorem is proved.

From (12) and (14).

$$(20) \quad \begin{aligned} s-{}_mS'_n &= {}^n[s+(s-s_1)+\ldots+(s-s_{m-1})]1/m^n \\ &= {}^n(s+s+\ldots+s)1/m^n - {}^n(0+s_1+\ldots+s_{m-1})1/m^n \\ &= s-{}^n(0+s_1+\ldots+s_{m-1})1/m^n \\ &= s-{}_mS_n \end{aligned}$$

where ${}_mS_n = (0+s_1+\ldots+s_{m-1})1/m^n$.

Then

(21) ${}_mS_n = {}_mS'_n$ or written out in full

$$(22) \quad \left(n\,s_1+\frac{n(n\pm1)^*}{2!}s_2+\ldots+s_{n(m-1)}\right)1/m^n$$

$$= [(m-1)a_1+(m-2)a_2+\ldots+a_{m-1}]1/m+$$

$$\sum_{i=1}^{n-1}\left[\frac{(m-1)\,{}^i(a_1+\ldots+a_m)+{}^i(a_2+\ldots+a_{m+1})+\ldots+{}^i(a_{m-1}+\ldots+a_{2m-2})}{m^{i+1}}\right]$$

Theorem II. When the series $a_1+a_2+a_3+\ldots$ converges to s

$$\lim_{n\to\infty} {}_mS_n = \lim_{m\to\infty} {}_mS_n = \lim_{m\to\infty} {}_mS'_n = \lim_{m,n\to\infty} {}_mS_n = \lim_{m,n\to\infty} {}_mS'_n = s.$$

That $\lim_{n\to\infty} {}_mS_n = s$ follows from (21) and Theorem I.

Consider $\lim_{m\to\infty} {}_mS_n = \lim_{m\to\infty} {}_mS'_n = s$. The proof of Theorem I suffices for these cases provided we fix n and let m vary, replace the conditions $n>N$ and $n>N'$ by $m>M$ and $m>M'$ and note in connection with (18) that the first k

* The plus sign occurs for $m>2$.

coefficients in the expansion of $(1+x+\ldots+x^{m-1})^n$ never change with m provided $m>K$.

Finally consider $\lim_{m,n\to\infty} {}_mS_n = \lim_{m,n\to\infty} {}_mS'_n = s$. Since n and m both increase, replace, in Theorem I, the conditions $n>N$ and $m>M'$ by the pairs $n>N$, $m>M$ and $n>N'$, $m>M'$, respectively, N' and M' having been chosen so that, $(m-1)n \geqq k$ and the inequality (18) holds for $n>N'$ and $m>M'$. This proof then suffices for the above cases.

Theorem III. When a series of variable terms converges uniformly it is uniformly summable by ${}_mS_n$ and ${}_mS'_n$ with either n or m or both increasing.

The proofs of Theorems I and II suffice for this theorem when (16) is replaced by $(R_{k,i}) \leqq \epsilon/2$, $i=1, 2, 3, \ldots$, for all values of the independent variables on the intervals considered. This, of course, is just the definition of uniform convergence.

Thus it has been proven that the condition of consistency is satisfied and uniformly convergent series are uniformly summable by ${}_mS_n$ and ${}_mS'_n$ with m fixed and n increasing or with n fixed and m increasing or with both m and n increasing in any manner whatever, m and n being on the range, $(1, 2, 3 \ldots)$.

Upon the basis of ${}_mS_n$ we define the two following sets of means:

(A)

$$ {}_mS_n^{[1]} = \left[n\,s_1 + \frac{n(n\pm1)s_2}{2!} + \ldots + s_{n(m-1)} \right] 1/m^n = {}_mS_n $$

$$ {}_mS_n^{[2]} = \left[n\,{}_mS_1^{[1]} + \frac{n(n\pm1)}{2!}\,{}_mS_2^{[1]} + \ldots + {}_mS_{n(m-1)}^{[1]} \right] 1/m^n $$

$$ {}_mS_n^{[r]} = \left[n\,{}_mS_1^{[r-1]} + \frac{n(n\pm1)}{2!}\,{}_mS_2^{[r-1]} + \ldots + {}_mS_{n(m-1)}^{[r-1]} \right] 1/m^n $$

where m is fixed and n increases.

(B)

$$ {}_m^{[1]}S_n = \left[n\,s_1 + \frac{n(n\pm1)}{2!}\,s_2 + \ldots + s_{n(m-1)} \right] 1/m^n = {}_mS_n $$

$$ {}_m^{[2]}S_n = \left[n\,{}_1^{[1]}S_n + \frac{n(n\pm1)}{2!}\,{}_2^{[1]}S_n + \ldots + {}_{n(m-1)}^{[1]}S_n \right] 1/m^n $$

$$ {}_m^{[r]}S_n = \left[n\,{}_1^{[r-1]}S_n + \frac{n(n\pm1)}{2!}\,{}_2^{[r-1]}S_n + \ldots + {}_{n(m-1)}^{[r-1]}S_n \right] 1/m^n $$

where n is fixed and m increases.

Theorem IV. The means (A) and (B) satisfy the condition of consistency

for all values of the parameters m and r in (A) and n and r in (B).

Consider first the means (A). We have proved (Theorem I) that the above theorem is true when r=1.

Hence when

(23) $s_1, s_2, \ldots s_n \ldots$ converges to s so does

(24) ${}_mS_1^{[1]}, {}_mS_2^{[1]}, \ldots {}_mS_n^{[1]} \ldots$

Applying Theorem I to ${}_mS_n^{[1]}$, we see that the sequence

$${}_mS_1^{[2]}, {}_mS_2^{[2]}, \ldots {}_mS_n^{[2]} \ldots$$

converges to s whenever (24) does, hence whenever (23) does.

The theorem follows by induction.

The argument for (B) is the analogue of this since the theorem is proven for r=1 in Theorem II.

Corollary. If (A) or (B) sums a series for any value of r, it sums the series to the same sum for every greater value of r.

Theorem V. The means (A) and (B) sum uniformly convergent series uniformly.

This theorem has been proven (Theorem III) for r=1. The proof for any r is the same as the proof of Theorem IV, except that the convergence of each sequence is uniform.

The means (A) and (B) can not sum properly divergent series.

This becomes evident for r=1 when we observe that for every k the sum of the last n(m—1)+1—k coefficients of $(1+x+\ldots+x^{m-1})^n$ divided by m^n, the sum of all of them, has the limit unity. The arguments for ${}_mS_n^{[r]}$ and ${}^{[r]}{}_mS_n$ are the same as that of Theorem IV except that we have proper divergence of each sequence instead of convergence.

If n and m increase together, in any manner whatever, in (A) and (B), Theorem IV, its corollary, Theorem V and the above remark on proper divergence hold for these methods. This follows from Theorem II and Theorem III by arguments similar to those used for (A) and (B).

Euler's Transformation for Slowly Convergent Series, with $x=-1$,* is a special instance of ${}_mS'_n$ namely ${}_2S'_n$, hence is identically equal to ${}_2S^{[1]}$, (${}^{[1]}{}_2S_n$). For putting m=2 in (22) gives

(25)

$$ {}_2S_n = (n\,s_1 + \frac{n(n-1)}{2!}s_2 + \ldots + s_n)\frac{1}{2^r} = \frac{a_1}{2} + \frac{a_1+a_2}{2^2} + \frac{a_1+2a_2+a_3}{2^3} + \ldots + $$

$$ \frac{a_1 + (n-1)a_2 + \frac{(n-1)(n-2)}{2!}a_3 + \ldots + a_n}{2^n} = S'. $$

* L. D. Ames, Annals of Mathematics, Series 2, Vol. 3, p. 188. Equation (1) of this article should read $s=u_1+u_2+u_3+\ldots$

The right member, which is Euler's form, is thus expressed in terms of $s_1, s_2, s_3 \ldots$

A Mean Having Frohenius' and Hölder's Means as Special Cases

If we add $\dfrac{a_1+a_2+\ldots+a_m}{m}$ to

$$ {}_mS_1, \left(= \frac{s_1+s_2+\ldots+s_{m-1}}{m}, \text{ we have } \frac{s_1+s_2+\ldots+s_m}{m} \right.$$

which is Frohenius' mean.† (When ${}_mS_1$ or Frohenius' mean has a limit, the other one has the same limit, for $ {}_mS_1 = \dfrac{s_1+s_2+\ldots+s_{m-1}}{m-1} \dfrac{m-1}{m} \Big)$. In general if we add $\dfrac{n(a_1+\ldots+a_m)}{m^n}$ to ${}_mS_n$ and form a set similar to (B) upon the basis of this sum we have a generalization of both Frohenius' and Hölder's ‡ means.

$$ {}_mS_n = \left[n\, s_1 + \frac{n(n\pm1)}{2!} s_2 + \ldots + s_{n(m-1)} \right] 1/m^n $$

Add $$ \left[a_1 + n\, a_2 + \frac{n(n\pm1)}{2!} a_3 + \ldots + a_{n(m-1)+1} \right] 1/m^n $$

and denote the sum by ${}_m(S)_n$.

Then

$$ (26) \quad {}_m(S)_n = \left[s_1 + n\, s_2 + \frac{n(n\pm1)}{2!} s_3 + \ldots + s_{n(m-1)+1} \right] 1/m^n. $$

Analogous to (B) we formulate:

(C)

$$ {}^{(1)}_{m}(S)_n = {}^n[s_1+\ldots+s_m]\, 1/m^n = {}_m(S)_n $$
$$ {}^{(2)}_{m}(S)_n = {}^n[{}^{(1)}_{1}(S)_n + \ldots + {}^{(1)}_{n}(S)_n]\, 1/m^n $$
$$ {}^{(r)}_{m}(S)_n = {}^n[{}^{(r-1)}_{1}(S)_n + \ldots + {}^{(r-1)}_{m}(S)_n]\, 1/m^n $$

where n is constant and m increases.

Theorem VI. The means (C) satisfy the condition of consistency and sum uniformly convergent series uniformly.

† Bromwich: Loc. cit., Article 51, Ex. 2.
‡ Bromwich: Loc. cit., Article 123.

From (21)

$$_m(S)_n = {}_mS'_n + {}^n(a_1 + \ldots + a_m)1/m^n = {}_m(S')_n, \text{ say,}$$

Then we can write

$$s - {}_m(S)_n = s - {}_m(S')_n$$

or

$$\frac{{}^n(s + \ldots + s)}{m^n} - {}_m(S)_n = s - {}_m(S')_n$$

whence

$${}^n[(s - s_1) + \ldots + (s - s_m)]1/m^n = s - {}_m(S')_n$$

or

$$(27) \qquad {}^n[R_1 + \ldots + R_m]1/m^n = s - {}_m(S')_n$$

Now, upon the basis of (27) results for ${}_m(S')_n$ and ${}_m(S)_n$ analogous to Theorem I, the first part of Theorem II and Theorem III are proven just as these were. Theorems for (C) analogous to Theorems IV and V then follow.

When n=1, (C) becomes,

$$\begin{aligned} {}^{[1]}_{m}(S)_1 &= [s_1 + s_2 + \ldots + s_m]1/m \\ {}^{[2]}_{m}(S)_1 &= [{}^{[1]}_{m}(S)_1 + {}^{[1]}_{m}(S)_1 + \ldots + {}^{[1]}_{m}(S)_1]1/m \\ {}^{[r]}_{m}(S)_1 &= [{}^{[r-1]}_{m}(S)_1 + {}^{[r-1]}_{m}(S)_1 + \ldots + {}^{[r-1]}_{m}(S)_1]1/m \end{aligned}$$

which is exactly Hölder's mean. Hölder, however, used $T^{[r]}$ where we are using ${}^{[r]}_{m}(S)_1$.

When n=r=1, in (C), we have Frobenius' mean.

There is an interesting relation between ${}_2(S)_n$ and Borel's original definition,* consequently between the latter and Euler's Transformation. Borel's original definition is

$$(28) \qquad \lim_{x\to\infty} \lim_{n\to\infty} \sum_{n}^{n} e^{-x} s_{n+1} \frac{x^n}{n!}$$

$$(29) \qquad {}_2(S)_n = \left[\left| s_1 + n s_2 + \frac{n(n-1)}{2.!} s_3 + \ldots + s_{n+1} \right| \right] 1\ 2^n$$

Introduce the factors $1, \frac{x}{n}, \frac{x^2}{n^2}, \ldots \frac{x^n}{n^n}$ as multipliers of the successive terms of (29) and replace $1/2^n$ by $\frac{1}{\left(\left(1 + \frac{x}{n}\right)^n\right)}$. We then have,†

* Bromwich: Loc. cit., Art. 114.

† This suggests a generalization of Borel's original definition upon the basis of (C), but the resulting forms are exceedingly complex.

$$(30)\quad \left[s_1+n/n\, s_2x+\frac{n(n-1)}{n^2}s_3\,x^2/2!+\ldots+s_{n+1}\,x^n/n^n\right]\frac{1}{\left((1+\frac{x}{n})\right)^n}$$

Denote this form by s (x,n).

Then,

$$\lim_{x\to\infty}\lim_{n\to\infty} s(x,n)=\lim_{x\to\infty}[s_1+s_2+s_3\,x^2/2!+\ldots]e^{-x}$$

$$=\lim_{x\to\infty}\lim_{n\to\infty}\sum_{o}^{n} e^{-x}s_{n+1}\,x^n/n!$$

while

$$\lim_{x=n\to\infty} s(x,n)=\lim_{n\to\infty} {}_2(S)_n$$

Thus the sum by Borel's form and that by ${}_2(S)_n$, when they exist, can be derived from the same function, s(x,n), the difference being in the manner of taking the limits.

Comparing (29) with (25) we see that

$${}_2(S)_n=\frac{a_1}{2}+\frac{a_1+a_2}{2^2}+\frac{a_1+2a_2+a_3}{2^3}+\ldots+\frac{{}^{n-1}(a_1+a_2)}{2^n}+\frac{{}^{n}(a_1+a_2)}{2^n}$$

That is, ${}_2(S)_n-\frac{{}^{n}(a_1+a_2)}{2^n}$ is Euler's Transformation. Whenever the latter converges to a limit, $\frac{{}^{(n-1)}(a_1+a_2)}{2^n}$ converges to zero, hence $\frac{{}^{n}(a_1+a_2)}{2^{n+1}}$ does. Consequently $\frac{{}^{n}(a_1+a_2)}{2^n}$ converges to zero. Hence when Euler's Transformation gives a sum, it is the same as $\lim_{x=n\to\infty} s(x,n)$.

The means, (A), sum series for which (B) fails just as Euler's method sums certain series for which Hölder's and Cesaro's methods fail. However, it is desirable to have different methods to use on different types of series provided they do not give contradictory results.*

Illustrations of the Use of (A) and (B) on Some Familiar Series

* See (b) p. 4.

(1) The series $1+x+x^2+x^3\ldots$ which is summable by (B) or Cesaro's or Hölder's methods only for $1>x\geq -1$ is summable by (A) for $x<1$. The actual work follows:

$$s_1=1=\frac{1-x}{1-x},\quad s_2=1+x=\frac{1-x^2}{1-x},\quad \ldots\quad s_n=\frac{1-x^n}{1-x},\qquad x\neq 1$$

Use (A) with m=2.

$$ {}_2S_n^{(1)}=\frac{1}{1-x}\left[n(1-x)+\frac{n(n-1)}{2}(1-x^2)+\ldots+(1-x^n)\right]\frac{1}{2^n}$$

$$=\frac{1}{1-x}\left[2^n-1-\left\{(1+x)^n-1\right\}\right]\frac{1}{2^n}=\frac{1}{1-x}\left[1-\frac{(1+x)^n}{2^n}\right]$$

$$ {}_2S_n^{(2)}=\frac{1}{1-x}\left[(n+\frac{n(n-1)}{2!}+\ldots+1)-\left\{\frac{n(1+x)}{2}+\frac{n(n-1)}{2!}\frac{(1+x)^2}{2^2}+\ldots\right.\right.$$

$$\left.\left.+\frac{(1+x)^n}{2^n}\right]\frac{1}{2^n}=\frac{1}{1-x}\left[2^n-1-\left\{(1+\frac{1+x}{2})^n-1\right\}\right]\frac{1}{2^n}=\frac{1}{1-x}\left[1-\frac{(3+x)^n}{2^{2n}}\right]\right.$$

Similarly

$$ {}_3S_n^{(1)}=\frac{1}{1-x}\left[2^n-1-\left\{\left(1+\frac{3+x}{2^2}\right)^n-1\right\}\right]\frac{1}{2^n}=\frac{1}{1-x}\left[1-\frac{(7+x)^n}{2^{3n}}\right]$$

$$ {}_3S_n^{(2)}=\frac{1}{1-x}\left[1-\frac{(15+x)^n}{2^{4n}}\right]$$

$$ {}_3S_n^{(3)}=\frac{1}{1-x}\left[1-\frac{(2^4-1+x)^n}{2^{4n}}\right]=\frac{1}{1-x}\left[1-\left(\frac{2^4-1+x}{2^4}\right)^n\right]$$

whence

$\lim_{n\to\infty} {}_3S_n^{(3)}=\frac{1}{1-x}$, $1>x>(1-2^{r+1})$, where r is arbitrary. That is, the sum of $1+x+x^2+\ldots$ is $\frac{1}{1-x}$ for every value of $x<1$.

Formulas (A) and (B) lend themselves readily to the actual calculation of

sums because of the many known relations between the coefficients in the expansion of a power of a polynomial.

(2) For instance all series which have

$$S_1=a^k,\ S_2=-(2a)^k,\ S_3=(3a)^k,\ S_4=-(4a)^k,\ \ldots$$

have the sum zero.

Using ${}_mS_n^{(1)}$ with m=2,

$${}_2S_n^{(1)}=\left[n\,a^k-\frac{n(n-1)}{2!}(2a)^k+\ldots+(-1)^{n+1}(n\,a)^k\right]\frac{1}{2^n}$$
$$=0,\ n>K^*$$

For this binomial form results from taking the nth mean of

$$0,\ a^k,-(2a)^k,\ \ldots,\ (-1)^{n+1}(n\,a)^k \qquad \text{(see pp. 6-7).}$$

a=1, K=1, gives the series

(2′) $$1-3+5-7+\ldots=0$$

a=1, K=2 gives the series

(2″) $$1-5+13-25+\ldots=0$$

(3) The series

1—1+1—1+.....

sums to 1/2 by (A) using any m or by (B) using any n, or by either with m and n both increasing.

Here

$$S_1=1,\ S_2=0,\ S_3=1,\ \ldots$$

$${}_mS_n^{(1)}=\left[n+0+\ldots+\frac{(1+(-1)^{n(m-1)+1}}{2}\right]1/m^n={}_mS_n$$

When m is even the sum of the even coefficients in $(1+x+\ldots+x^{m-1})^n$ equals the sum of the odd, and when m is odd, the sum of the even equals the sum of the odd minus unity. In either case if O and E denote the sum of the odd and of the even coefficients, respectively,

$$E+O=2E+\frac{1+(-1)^{m+1}}{2}$$

That is,

$$m^n=2E+\frac{1+(-1)^{m+1}}{2}$$

---

* A. M. Kenyon: "Some Properties of Binomial Coefficients," Proceedings of the Indiana Academy of Science, 1914, p. 2.

Whence

$$E=\frac{m^n}{2}-\frac{1+(-1)^{m+1}}{4}$$

Substituting this value of E in the above form of S,

$${}_mS_n=\frac{1}{2}\left[m^n-\frac{1+(-1)^{m+1}}{2}\right]1/m^n$$

Hence

$$\lim_{n\to\infty}{}_mS_n=\lim_{m\to\infty}{}_mS_n=\lim_{m,n\to\infty}{}_mS_n=\frac{1}{2}$$

(4) The series

$$a-(a+d)+(a+2d)-(a+3d)+\ldots$$

sums to $\frac{2a-d}{4}$.

$$s_1=a,\quad s_2=d,\quad s_3=a+d,\quad s_4=-2d,\quad s_5=a+2d,\ \ldots$$

$${}_2S_n=\left[n\,a-\frac{n(n-1)}{2!}d+\frac{n(n-1)(n-2)}{3!}(a+d)-\frac{n(n-1)(n-2)(n-3)}{4!}2d\right.$$

$$\left.+\frac{n\ldots(n-4)}{5!}(a+2d)-\ldots\right]\frac{1}{2^n}$$

$$=\left[\frac{1}{2}a-2^n-d\left\{\frac{n(n-1)}{2!}-\frac{n(n-1)(n-2)}{3!}+\frac{n(n-1)(n-2)(n-3)}{4!}2\right.\right.$$

$$\left.\left.-\frac{n\ldots(n-4.}{5!}2\ldots\right\}\right]\frac{1}{2^n}$$

Simplify the coefficient of a. In the coefficient of d add and subtract $\frac{n}{2}$.

Then collect one-half of the sum of the even binomial coefficients.

This gives,

$${}_2S_n=\frac{2}{2}-\frac{d}{2^n}\left[\frac{n}{2}+\frac{1}{2}\frac{n(n-1)(n-2)}{3!}+\ldots+\frac{1}{2}\right]-\frac{d}{2^n}\left[\frac{-1}{2}n+\frac{2}{2}\frac{n(n-1)}{2!}\right.$$

$$\left.-\frac{3}{2}\frac{n(n-1)(n-2)}{3!}+\frac{4}{2}\frac{n\ldots(n-3)}{4!}-\frac{5}{2}\frac{n\ldots(n-4)}{5!}+\ldots+(-1)^n\frac{n}{2}\right]$$

$$= \frac{a}{2} - \frac{d}{2^n}\frac{2^n}{2^2} - \frac{d}{2^{n+1}}\left[0 - n + 2\frac{n(n-1)}{2!} - 3\frac{n(n-1)(n-2)}{3!}\right.$$

$$\left. + 4\frac{n\ldots(n-3)}{4!} - \ldots + (-1)^n n\right]$$

The bracket is zero, $n \geqq 2$, since it results from taking the nth mean two at a time of the terms 0, 1, —2, 3, —4, 5, ..., $(-1)^n n$, (see pp. 4-6).

Whence,

$$_2S_n = \frac{2a-d}{4} = \lim_{n\to\infty} {}_2S_n$$

(4′) $a=1, d=1$ gives

$$1-2+3-4+5\ldots = \tfrac{1}{4}$$

(4″) $a=1, d=3$ gives

$$1-4+7-10+\ldots = -\tfrac{1}{4}$$

It is interesting to notice that the sum of

$$a-(a+d)+(a+2d)-(a+3d)+\ldots$$

is positive, zero or negative according as $d \lessgtr 2a$. (See (2′), p. 16.)

This result could have been obtained by using (B).

3. **Non-summable Series.***

Consider the series

(1) $$a_1+a_2+a_3+\ldots+a_n+\ldots$$

and the sequence

(2) $$s_1, s_2, s_3, \ldots, s_n, \ldots \text{ where } s_n = \sum_1^n a_p$$

Definition. Denote by N_s the class of series which are such that for every c and every M there exists an $m \geqq M$ such that for every $n \geqq 1$,

$$\sum_{p=1}^{n} (s_{m+p} - c) \geqq 0, \text{ (or } \leqq 0).$$

The class N_s contains all properly divergent series and such series † as

* Presented to the American Mathematical Society, April 29, 1916.

† The sequences of S's are respectively: 1, 0, 2, 0, 3, 0, 4, 0, ... and 2, —7, 4, —2, 6, —3, ... In either case for a given C we can choose m such that $S_{m-1}+S_{m-2} > C$.

consequently $\sum_{p=1}^{n} (S_{m+p} - C) \geqq 0$, $n \geqq 1$.

$$1-1+2-2+3-3+\ldots+(-1)^{n+1}\frac{2n+1+(-1)^{n+1}}{4}+\ldots$$

$$2-3+5-6+8-9+\ldots+(-1)^{n+1}\frac{6n+1+(-1)^{n+1}}{4}\ldots$$

The method of procedure in this article is to place sufficient restrictions upon the general definition, (1), § 1, to make the members of N_1 nonsummable.‡, then prove upon the basis of this general method, that the specific methods mentioned in § 1 can not sum series in the class N_1. It is, incidentally, shown that the general definition thus restricted satisfies the condition of consistency and sums uniformly convergent series uniformly.

Definition (1) § 1 can be expressed in terms of $s_1, s_2, \ldots s_n \ldots$ as follows:

(3) $S_n=f_1a_1+f_2a_2+\ldots+f_{n-1}a_{n-1}+f_na_n.$

(3′) $=f_1s_1+f_2(s_2-s_1)+\ldots+f_{n-1}(s_{n-1}-s_{n-2})+f_n(s_n-s_{n-1})$

(3″) $=s_1(f_1-f_2)+s_2(f_2-f_3)+\ldots+s_{n-1}(f_{n-1}-f_n)+s_nf_n$

(4) $=s_1\Phi_1+s_2\Phi_2+\ldots+s_{n-1}\Phi_{n-1}+s_nf_n$

where

$$f_p-f_{p+1}=\Phi_p, \qquad p=1, 2, 3, \ldots, n-1,$$

whence

$$(5)\quad S=\lim_{x_i\to L_i}\lim_{n\to\infty}\left[\sum_{1}^{n-1}s_p\Phi_p(x_i)+s_nf_n\right], \qquad i=1, 2, 3, \ldots k$$

Now let

(6) $\Theta_p=s_p-c$, (see definition, p. 18), and replace s_p by $c+\Theta_p$ in

$$S_n=\sum_{1}^{n-1}s_p\Phi_p+s_nf_n$$

Then

$$(7)\qquad S_n=c\sum_{1}^{n-1}\Phi_p+\sum_{1}^{n-1}\Theta_p\Phi_p+s_nf_n$$

But

$$\sum_{1}^{n-1}\Phi_p=\sum_{1}^{n-1}(f_{p-1}-f_p)=f_1-f_n$$

‡ These restrictions are chosen so as to include in the restricted general method those special methods by which the class of series N_1 are non-summable, these specific results having been obtained independently.

Whence

$$S_n = c f_1 + \sum_{1}^{n-1} \Theta_p \Phi_p + (s_n - c) f_n$$

or

$$S_n = c f_1 + \sum_{1}^{m} \Theta_p \Phi_p + \sum_{m+1}^{n-1} \Theta_p \Phi_p + \Theta_n f_n \tag{8}$$

where m is arbitrary and $n \geqq m+2$.

Consider, first, the terms

$$c f_1 + \sum_{1}^{m} \Theta_p \Phi_p$$

If we assume

$$\lim_{x_i \to L_i} f_p(x_i) = 1, \ (i = 1, 2, \ldots, k), \text{ for every } p, \tag{9}$$

then

$$\lim_{x_i \to L_i} \Phi_p = \lim_{x_i \to L_i} (f_{p-1} - f_p) = 0$$

whence

$$\lim_{x_i \to L_i} (c f_1 + \sum_{1}^{m} \Theta_p \Phi_p) = c$$

Now consider the term $\sum_{m+1}^{n-1} \Theta_p \Phi_p$

Let $s'_n = \sum_{m+1}^{n-1} \Theta_p$. Consider c positive and $\sum_{m+1}^{n-1} \Theta_p \geqq 0$ for $n > m+1 \geqq M$.

(See definition, p. 18.)

Then in the light of (4) we can write

$$\sum_{m+1}^{n-1} \Theta_p \Phi_p = s'_{m+1}(\Phi_{m+1} - \Phi_{m+2}) + s'_{m+2}(\Phi_{m+2} - \Phi_{m+3}) + \ldots \tag{10}$$

$$+ s'_{n-2}(\Phi_{n-2} - \Phi_{n-1}) + s'_{n-1}\Phi_{n-1}$$

and $\sum_{m+1}^{n-1} \Theta_p \Phi_p$ will be positive or zero provided $(\Phi_{m+p} - \Phi_{m+p+1})$,

$(p = 1, 2 \ldots, (n - m - 2)$ and Θ_{n-1}, are positive or zero. This will be the case if we assume that

(11) Φ_{m+p}, $[=f_{m+p}(x_i)-f_{m+p-1}(x_i)]$, $p \geqq 0$, never increases with p, (for any set of values of x_i, $i=1, 2, 3, \ldots k$), M having been properly chosen, and that

(12) $\lim_{n\to\infty} f_n(x_i)=0$, for all values of x_1.

For from (12) $\lim_{n\to\infty} \Phi_n=0$ and from this fact and (11),

(13) $\Phi_{n-1}=(f_{n-1}-f_n) \geqq 0, \qquad n \geqq M+1$

Finally consider the last term of (8), namely $\Theta_n f_n$. Evidently

(14) $f_n(x_i) \geqq 0, \qquad n \geqq M+1.$

Hence $\Theta_n f_n$ can not remain negative, for Θ_n can not remain negative and increase without limit in absolute value since $\sum_{p=1}^{n} \Theta_{m+p} \geq 0, \qquad n > 1.$

We have thus shown that S_n either oscillates or can be made greater than any positive quantity C, by a proper choice of m.

The argument for series such that $\sum_{p=1}^{n} (s_{m+p}-c) \leqq 0$, is the same as the above.

Theorem VII. Every series of the class N_x is non-summable by the definition

$$S=\lim_{x_i\to L_i} \lim_{n\to\infty} \sum_{p=1}^{n} a_p f_p(x_i), \qquad i=1, 2, \ldots k,$$

where

I. $\lim_{x_i\to L_i} f_p(x_i)=1$, for every p

II. $\lim_{n\to\infty} f_n(x_i)=0$

and

III. For every set of values of $x_1, x_2, \ldots, x_k$, there exists an M such that the sequence,

$$(f_{m+p}-f_{m+p+1}), \quad p=0, 1, 2, \ldots$$

never increases.

Theorem VIII. Definition (1) § 1 under restrictions I, II and III satisfies the condition of consistency.

Let $a_1+a_2+\ldots+a_n+\ldots$ converge.

Since, $s-R_1=s_1,\ s-R_2=s_2,\ \ldots,\ s-R_n=s_n$

where

$$R_i=a_{i+1}+a_{i+2}\ldots$$

the form of s_n in (3″) (p. 19), can be written, after collecting the coefficients of s,

$$S_n=sf_1-\sum_{p=1}^{n-1} R_p(f_p-f_{p+1})-R_nf_n$$

and for any m, ($\leqq$n—1), we can write

$$S_n=s\ f_1-\sum_{1}^{m} R_p(f_p-f_{p+1})-\left[\sum_{m+1}^{n-1} R_p(f_p-f_{p+1})+R_nf_n\right]$$

By I of Theorem VII,

$$\lim_{x_i\to L_i}\left[s\,f_1-\sum_{1}^{m} R_p(f_p-f_{p+1})\right]=s$$

The last term on the right converges to zero for

$$\left[\sum_{m+1}^{n-1} R_p(f_p-f_{p+1})+R_nf_n\right]$$

$$<\epsilon\left[f_{m+1}-f_{m+2}+f_{m+2}-\ldots+f_n\right]$$

provided m is chosen large enough to make III, Theorem VII, hold and also to make $|R_p|<\epsilon,\ p>m$.

Now from II and III, there exists an m such that *

$$o\leqq f_p<1, \qquad p>m$$

whence there exists an m such that

$$\left[\sum_{m+1}^{n-1} R_p(f_p-f_{p+1})+R_nf_n\right]<\epsilon, \qquad n>m+1.$$

Corollary. The general definition restricted by I, II and III sums uniformly convergent series uniformly.

Theorem IX. Le Roy's method can not sum series in the class N_s.

Le Roy's † definition is

$$(15)\qquad S=\lim_{t\to 1}\lim_{n\to\infty}\sum_{p=o}^{n}\frac{\Gamma(pt+1)}{\Gamma(p+1)}a_p, \qquad 1>t>o.$$

* Choose an m such that $|f_{m+p}|<1,\ p\geqq o$. Then the sequence, $f_m, f_{m+1}, \ldots$, never increases and converges to zero according to II and III. Hence $f_{m+p}>o$.

† Annales de Toulouse, Ser. 2, Vol. 2 (1900), pp. 323-327.

Comparing with the general definition of Theorem VII we see that

$$f_p(x_1, x_2, \ldots, x_k)=f_p(t)=\frac{\Gamma[(p-1)t+1]}{\Gamma(p)}$$

Condition I of the above theorem is satisfied since

$$\lim_{t\to 1} \frac{\Gamma[(p-1)t+1]}{\Gamma(p))}=1$$

Condition II holds for $\lim_{n\to\infty} \frac{\Gamma[(n-1)t+1]}{\Gamma(n)}=0^*,\ 1>t>0.$

In order to show that Condition III holds it must be shown that for every $t,\ 1>t>0$, there exists a P such that for every $p>P$,

$$(16)\quad \frac{\Gamma(pt+1)}{\Gamma(p+1)}-\frac{\Gamma(pt+1+t)}{\Gamma(p+2)}\leqq\frac{\Gamma(pt+1+t)}{\Gamma(p+2)}-\frac{\Gamma(pt+1+2t)}{\Gamma(p+3)}$$

Since p is a positive integer, the denominators are, respectively, $p!$, $(p+1)!$ and $(p+2)!$ By the identity $\Gamma(x+1)=x\Gamma(x)$,

$$\Gamma(pt+1+t)=(p+1)t\Gamma(pt+t),$$

and

$$\Gamma(pt+1+2t)=(p+2)t\Gamma(pt+2t)$$

whence (16) reduces to

$$\Gamma(pt+1)-t\Gamma(pt+t)\leqq t\Gamma(pt+t)-\frac{t}{p+1}\Gamma(pt+2t)$$

or

$$\frac{\Gamma(pt+1)}{\Gamma(pt+t)}-t\leqq t\left[1-\frac{\Gamma(pt+2t)}{(p+1)\Gamma(pt+t)}\right]$$

$$\leqq t\left[1-t\frac{\Gamma(pt+2t)}{\Gamma(pt+1+t)}\right]$$

Since the right hand side is less than t it will be sufficient to show that

$$\frac{\Gamma(pt+1)}{\Gamma(pt+t)}\leqq 2t$$

* Small: Dissertation, Columbia University, 1913, pp. 38-39.

By Sterlings formula,*

$$(17)\qquad \lim_{p\to\infty} \frac{\Gamma(pt+1)}{e^{-pt}(pt)^{pt+1/2}\sqrt{2\pi}} \cdot \frac{e^{-(pt+t-1)}(pt+t-1)^{pt+t-1/2}\sqrt{2\pi}}{\Gamma(pt+t)} = 1$$

Or simplifying,

$$(18)\qquad \lim_{p\to\infty} \frac{\Gamma(pt+1)}{\Gamma(pt+t)} e^{1-t}\left(1+\frac{t-1}{pt}\right)^{pt+1/2}(pt+t-1)^{t-1} = 1$$

But

$$\lim_{p\to\infty} \left(1+\frac{t-1}{pt}\right)^{pt-1/2} = e^{t-1}$$

and

$$\lim_{p\to\infty} (pt+t-1)^{t-1} = 0, \text{ since } t<1.$$

Thus the limit of the coefficient of $\frac{\Gamma(pt+1)}{\Gamma(pt+t)}$ in (18) is zero.

Hence,

$\frac{\Gamma(pt+1)}{\Gamma(pt+t)}$ must increase without limit as p increases,

which proves (16).

Thus LeRoy's method has been shown to be a special case of the general definition restricted as in Theorem VII. Hence both Theorem VII and Theorem VIII † hold for it.

Theorem X. Neither Borel's integral definition $S=\int_0^\infty e^{-x}u(x)dx$ nor his original definition $S=\lim_{x\to\infty} \sum_0^\infty S_p \frac{e^{-x}x^p}{p!}$ can sum series of the class N_σ.

If Borel's original definition sums a series, his integral definition gives the same sum.‡ and if Borel's integral definition sums a series, LeRoy's method gives the same sum. § By Theorem IX, LeRoy's method can not sum series of the class N_σ, hence neither of Borel's methods can.¶

---

* Bromwich: Loc. cit., page 462.

† See note, page 28.

‡ Bromwich: Loc. cit., Article 114.

§ Bromwich: Loc. cit., Article 116.

¶ It can be shown by an argument similar to that of Theorem IX that Borel's methods are special instances of the general method, see note, p. 26.

Theorem XI. Cesaro's mean * $s=\lim_{n\to\infty}\frac{S_n^{(r)}}{A_n^{(r)}}$,

where $S_n^{(r)}=s_n+r\,s_{n-1}+\frac{r(r+1)}{2!}S_{n-2}+\ldots+\frac{r(r+1)\ldots(r+n-1)}{n!}s_0$

and $A_n^{(r)}=(r+1)(r+2)\ldots(r+n)/n!$,

can not sum series of the type N_s.

First consider $r\leqq 1$.

S_n can be written

$$(19)\quad a_0+\frac{n}{r+n}a_1+\frac{n(n-1)}{(r+n-1)(r+n)}a_2+\ldots+\frac{n!a_n}{(r+1)(r+2)\ldots(r+n)}$$

It is convenient, now, to let p run from 0 to n, instead of 1 to n, in the general definition of Theorem VII. This complies with the usage in Cesaro's mean. Then,†

$$(20)\quad f_p(x_i)=f_p(n)=\frac{n(n-1)\ldots(n-p+1)}{(r+n-p+1)(r+n-p+2)\ldots(r+n)},\qquad p\leqq n$$

Now

$f_p(n)$ may be written,

$$\frac{n}{r+n-p+1}\cdot\frac{n-1}{(r+n-p+2)}\cdots\frac{n-p+1}{r+n}$$

Each of these fractions approaches unity as n increases, and there are exactly p of them. Hence,

$$(21)\qquad \lim_{n\to\infty} f_p(n)=1,\text{ for every p.}‡$$

that is, condition I of Theorem VII holds.

Again,

$$f_n(n)=\frac{n!}{(r+1)(r+2)\ldots(r+n)}=\frac{2}{r+1}\cdot\frac{3}{r+2}\cdots\frac{n}{r+n-1}\cdot\frac{1}{n+r}$$

* Bullitin des Sciences Math. (2), V. 14 (1890), p. 119.

† In this case $x_k=n$. See note, p. 1.

‡ In this mean $f_0(n)=1$.

Every fraction on the right is equal to or less than unity and $\frac{1}{n+r}$ has the limit zero. Hence

$$\lim_{n\to\infty} f_n(n)=0, \tag{22}$$

that is, condition II is satisfied.

Condition III. From (19) it is easily seen that f_p-f_{p+1}, $p=0, 1, 2, \ldots$, is,

$$\frac{r}{r+n}, \frac{rn}{(r+n-1)(r+n)}, \frac{rn(n-1)}{(r+n-2)(r+n-1)(r+n)}\cdots$$

These terms never increase when $r \geqq 1$.

Finally it is necessary to note that f_p-f_{p+1} is always positive since (13), page 21, does not, in this case, follow from conditions II and III. It would, also, suffice to prove $\lim_{n\to\infty} f_{n-1}(n)=0$.

If Cesaro's mean sums a series with any value of r, it gives the same sum with any greater r.* Hence Cesaro's mean can sum series in N_s for no value of r.†

Theorem XII. Hölder's mean can not sum series of the class N_s.

It has been shown by Knopp ‡ that Hölder's mean

$$T_n^{(1)}=\frac{1}{n+1}(s_0+s_1+\ldots+s_n)=\frac{s_n^{(1)}}{A_n^{(1)}}$$

$$T_n^{(2)}=\frac{1}{n+1}(T_0^{(1)}+T_1^{(1)}+\ldots+T_n^{(1)})$$

$$T_n^{(r)}=\frac{1}{n+1}(T_0^{(r-1)}+T_1^{(r-1)}+\ldots+T_n^{(r-1)})$$

gives a sum only when Cesaro's gives the same sum. Thus the above theorem is a consequence of Theorem XI.

The means § (A), (B) and (C) of article 2 do not satisfy condition III of Theorem VII for the polynomial coefficients first increase and then decrease and the maximum coefficient moves out the sequence $s_1, s_2, \ldots$ as n increases. Hence we cannot choose an m such that $f_{m+p}-f_{m+p+1}$ never increases as p increases.

---

* Chapman, Proc.'s, London Math'l Soc., Vol 9 (1911), pp. 369-409.

† We have as a corollary of Theorems IX, X and XI, that LeRoy's, Borel's (see note, p. 24) and Cesaro's methods satisfy the condition of consistency and sum uniformly convergent series uniformly but these are familiar results.

‡ K. Knopp, Inaugural Dissertation (Berlin, 1907), p. 19.

§ The question of the non-summability of series in N_s by these means will be considered in a later paper.

BIBLIOGRAPHY

Ford, "Studies on Divergent Series and Summability," Michigan Science Series, Vol. II. This treatise contains an extensive bibliography to which I will only add,

L. L. Smail's Dissertation, "Some Generalizations in the Theory of Divergent Series," Columbia University, 1913.

VITA

Glenn James was born October 2nd, 1882; attended Vincennes University 1901-1903; received the A. B. degree from Indiana University 1905 and the A. M. 1910; studied at Chicago University two summers, and at Columbia one summer and throughout the year 1915-1916; instructor at the Michigan Agricultural College 1905-1908 and at Purdue University since then; member of The American Mathematical Society and the Indiana Academy of Science; has published "The Accuracy of Interpolation in a Five Place Table of Logarithms of Sines" in The American Mathematical Monthly, October, 1913, in conjunction with Professor A. M. Kenyon, and "A Note on the Sum of the Remainders of a Convergent Series," American Mathematical Monthly, November, 1916. He wishes to express his gratitude to Professor W. B. Fite for the inspiration received from him and his very careful criticism of this dissertation.